Gelfand and Kolmogorov numbers of Sobolev embeddings of weighted function spaces

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Abstract. In this paper we study the Gelfand and Kolmogorov numbers of Sobolev embeddings between weighted function spaces of Besov and Triebel-Lizorkin type with polynomial weights. The sharp asymptotic estimates are determined in the so-called non-limiting case.

Key words: Gelfand numbers; Kolmogorov numbers; Sobolev embeddings; Weighted function spaces.

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1 Introduction

In recent years a great deal of effort has gone into studying compactness of Sobolev embeddings between function spaces of Besov and Triebel-Lizorkin type from the standpoint of n-widths, especially approximation, Gelfand and Kolmogorov numbers. The case of function spaces defined on bounded domains has attracted a lot of attention, see [5, 21, 25, 28, 32]. For weighted function spaces of this type, such embeddings have also been studied by many authors, with polynomial (and more general) weights considered. Some breakthroughs on approximation numbers may be found in the works of Caetano [2], Haroske [9, 10], Haroske and Skrzypczak [11, 12], Mynbaev and Otel'baev [18], Piotrowska [22], Skrzypczak [24, 25] and Vasil'eva [31]. In particular, Skrzypczak [24] investigated the approximation numbers of the embeddings in the case of polynomial weights, by using operator ideals. In the context of Gelfand and Kolmogorov numbers, Vasil'eva [30, 31] established the asymptotics of the Kolmogorov numbers of weighted Sobolev classes on a finite interval or half-axis in the space L_q with weight. However, the estimates in many other cases are still left open.

n-Widths are a well-explored subject in approximation theory, see [19, 21, 32], and recently they have been applied in many areas, including compressed sensing [4], computational mechanics [1, 6] and spectral theory [5]. In particular, in the remarkable paper introducing compressed sensing [4], general performance bounds for sparse recovery methods are obtained by means of the theory of n-widths. In [1, 6] the Kolmogorov number is utilized to assess approximation

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properties of functions employed in finite element techniques. The performance of approximation numbers for describing the spectral properties of (pseudo-)differential operators is discussed in [5].

In this paper we present the sharp asymptotic estimates of the Gelfand and Kolmogorov numbers in the so-called non-limiting case. Although there are parallel considerations which cover some cases in Vasil'eva [30, 31], we proceed in a completely different way than Vasil'eva. We shall follow the method utilized for approximation numbers in Skrzypczak [24] with its corrigendum [26].

Motivated by [16, 24, 25], using the discretization method due to Maiorov [17], we reduce the function space of the problem to a weighted sequence space, then we determine the asymptotic behavior of the Gelfand and Kolmogorov numbers of Sobolev embeddings between weighted function spaces. The discretization technique is very important in the process of determining the exact order of n-widths of such classes, and in many cases it plays a major role. Moreover, our main tools are the use of operator ideals, see [3, 19, 20], and the basic estimates of related widths of the Euclidean ball due to Gluskin [8]. Historically, the technique of estimating single n-widths via estimates of ideal quasi-norms derives from ideas of Carl [3].

Following Skrzypczak [24], we concentrate on the spaces with polynomial weights

$$w_\alpha(x) := (1 + |x|^2)^{\alpha/2} \tag{1.1}$$

for some exponent $\alpha > 0$. Let

$$-\infty < s_2 < s_1 < \infty, \qquad 1 \le p_1 \le p_2 \le \infty, \qquad 1 \le q_1, q_2 \le \infty. \tag{1.2}$$

It is well known that if $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$ then

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d). \tag{1.3}$$

Moreover, the non-limiting case means as usual that $\delta \neq \alpha$. The case $p_2 < p_1$ is also considered in this article.

Remark 1.1. *In [13], it is proved that the embedding*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d, v_1) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d, v_2) \tag{1.4}$$

(and its F-counterparts with $p_2 < \infty$) is compact if, and only if,

$$s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2} \quad \text{and} \quad \frac{v_2(x)}{v_1(x)} \to 0 \quad \text{for} \quad |x| \to \infty, \tag{1.5}$$

where $-\infty < s_2 < s_1 < \infty, 0 < p_1 \le p_2 \le \infty, 0 < q_1, q_2 \le \infty$ and v_1, v_2 are admissible weight functions, see also [9].

Based on these considerations (where comparatively general weight functions are involved), we can assume that the target space is an unweighted space. Therefore, we restrict ourselves to the so-called standard situation: $v_1(x) = w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$, $\alpha > 0$, $v_2(x) \equiv 1$.

Notation 1.2. *By the symbol '$\hookrightarrow$' we denote continuous embeddings.*

Identity operators will always be denoted by id. *Sometimes we do not indicate the spaces where* id *is considered, and likewise for other operators.*

Let X and Y be complex Banach spaces and denote by $\mathcal{L}(X,Y)$ the class of all linear continuous operators $T : X \to Y$. If no ambiguity arises, we write $\|T\|$ instead of the more exact versions $\|T \mid \mathcal{L}(X,Y)\|$ or $\|T : X \to Y\|$.

The symbol $a_n \sim b_n$ means that there exists a constant $c > 0$ independent of n such that

$$c^{-1} a_n \le b_n \le c a_n, \qquad n = 1, 2, 3 \ldots .$$

All unimportant constants will be denoted by c or C, sometimes with additional indices.

We start with recalling the definitions of Kolmogorov and Gelfand numbers, cf. [21]. We use the symbol $A \subset\subset B$ if A is a closed subspace of a topological vector space B.

Definition 1.3. *Let $T \in \mathcal{L}(X,Y)$.*

(*i*) *The nth Kolmogorov number of the operator T is defined by*

$$d_n(T, X, Y) = \inf\{\|Q_N^Y T\| :\, N \subset\subset Y,\, \dim(N) < n\},$$

also written by $d_n(T)$ if no confusion is possible. Here, Q_N^Y stands for the natural surjection of Y onto the quotient space Y/N.

(*ii*) *The nth Gelfand number of the operator T is defined by*

$$c_n(T, X, Y) = \inf\{\|T J_M^X\| :\, M \subset\subset X,\, \operatorname{codim}(M) < n\},$$

also written by $c_n(T)$ if no confusion is possible. Here, J_M^X stands for the natural injection of M into X.

It is well-known that the operator T is compact if and only if $\lim_n d_n(T) = 0$ or equivalently $\lim_n c_n(T) = 0$, see [21].

The Kolmogorov and Gelfand numbers are both examples of so-called s-numbers, cf. [19, 20, 21]. Let s_n denote either of these two quantities, c_n or d_n, and let Y be a Banach space. We collect several common properties of Kolmogorov and Gelfand numbers below:

(PS1) (nonincreasing property) $\|T\| = s_1(T) \ge s_2(T) \ge \cdots \ge 0$ for all $T \in \mathcal{L}(X,Y)$,

(PS2) (subadditivity) $s_{m+n-1}(S+T) \le s_m(S) + s_n(T)$ for all $m, n \in \mathbb{N}$, $S, T \in \mathcal{L}(X,Y)$,

(PS3) (multiplicativity) $s_{m+n-1}(ST) \le s_m(S) s_n(T)$ for all $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$ and $m, n \in \mathbb{N}$, cf. [19, p. 155], where Z denotes a Banach space,

(PS4) (rank property) $\operatorname{rank}(T) < n$ if and only if $s_n(T) = 0$, where $T \in \mathcal{L}(X,Y)$.

Both concepts, Kolmogorov and Gelfand numbers, are related to each other. Namely they are dual to each other in the following sense, cf. [19, 21]: If X and Y are Banach spaces, then

$$c_n(T^*) = d_n(T) \tag{1.6}$$

for all compact operators $T \in \mathcal{L}(X,Y)$ and

$$d_n(T^*) = c_n(T) \tag{1.7}$$

for all $T \in \mathcal{L}(X,Y)$.

Following Pietsch [20], we associate to the sequence of the Kolmogorov (or Gelfand) numbers the following operator ideals, and for $0 < r < \infty$, we put

$$\mathscr{L}^{(s)}_{r,\infty} := \left\{ T \in \mathcal{L}(X,Y) : \quad \sup_{n\in\mathbb{N}} n^{1/r} s_n(T) < \infty \right\}. \tag{1.8}$$

Equipped with the quasi-norm

$$L^{(s)}_{r,\infty}(T) := \sup_{n\in\mathbb{N}} n^{1/r} s_n(T), \tag{1.9}$$

the set $\mathscr{L}^{(s)}_{r,\infty}$ becomes a quasi-Banach space. For such quasi-Banach spaces there always exists a real number $0 < \rho \le 1$ such that

$$L^{(s)}_{r,\infty}\left(\sum_j T_j\right)^{\rho} \le C \sum_j L^{(s)}_{r,\infty}(T_j)^{\rho} \tag{1.10}$$

holds for any sequence of operators $T_j \in \mathscr{L}^{(s)}_{r,\infty}$. Then we shall use the quasi-norms $L^{(c)}_{r,\infty}$ and $L^{(d)}_{r,\infty}$ for the Gelfand and Kolmogorov numbers, respectively.

The paper is structured as follows. In Sect. 2, we introduce weighted function spaces of B-type and F-type, and provide our main results. In Sect. 3, the crucial part of the work will be done, we investigate the Kolmogorov numbers of embeddings of related sequence spaces. Finally, in Sect. 4, these results will be used to derive the desired Kolmogorov number estimates for the function space embeddings under consideration, and similar results on the Gelfand numbers of such embeddings are established. Our main assertions are Theorem 2.5 and Theorem 2.7.

2 Main results

We suppose that the reader is familiar with (unweighted) function spaces of B-type and F-type on $\mathbb{R}^d$. One can consult [5, 27] and many other literatures for the definitions and basic properties.

Throughout this paper we are interested in the function spaces with polynomial weights given by (1.1). As usual, $\mathcal{S}'(\mathbb{R}^d)$ denotes the set of all tempered distributions on the Euclidean d-space $\mathbb{R}^d$. For us it will be convenient to introduce weighted function spaces to be studied here.

Definition 2.1. *Let $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$. Then we put*

$$B^s_{p,q}(\mathbb{R}^d, w_\alpha) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f \mid B^s_{p,q}(\mathbb{R}^d, w_\alpha)\| = \|f w_\alpha \mid B^s_{p,q}(\mathbb{R}^d)\| < \infty \right\},$$

$$F^s_{p,q}(\mathbb{R}^d, w_\alpha) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|f \mid F^s_{p,q}(\mathbb{R}^d, w_\alpha)\| = \|f w_\alpha \mid F^s_{p,q}(\mathbb{R}^d)\| < \infty \right\},$$

with $p < \infty$ for the F-spaces.

Remark 2.2. *If no ambiguity arises, then we can write $B^s_{p,q}(w_\alpha)$ and $F^s_{p,q}(w_\alpha)$ for brevity.*

Remark 2.3. *There are different ways to introduce weighted function spaces, see, e.g., Edmunds and Triebel [5], or Schmeisser and Triebel [23]. One can also consult [15, 28] for related remarks.*

Let $A^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($A^s_{p,q}(\mathbb{R}^d)$) stand for either $B^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($B^s_{p,q}(\mathbb{R}^d)$) or $F^s_{p,q}(\mathbb{R}^d, w_\alpha)$ ($F^s_{p,q}(\mathbb{R}^d)$), with the constraint that for the F-spaces $p < \infty$ holds.

Now we give a necessary and sufficient condition for compactness of the embeddings under consideration, which was proved in [13], cf. also [5, 16].

Proposition 2.4. *Suppose $1 \le p_1, p_2, q_1, q_2 \le \infty$ and $-\infty < s_2 < s_1 < \infty$. Let $\alpha > 0$, $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2})$. The embedding $B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ is compact if and only if $\min(\alpha, \delta) > d \max(\frac{1}{p_2} - \frac{1}{p_1}, 0)$.*

A similar theorem also holds for $F^s_{p,q}$-spaces. We are now ready to formulate our main results.

Theorem 2.5. *Suppose $1 \le p_1, p_2, q_1, q_2 \le \infty$ and $-\infty < s_2 < s_1 < \infty$. Let $\alpha > 0$, $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$, $\theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2}$, and $\frac{1}{\tilde{p}} = \frac{\mu}{d} + \frac{1}{p_1}$, where $\mu = \min(\alpha, \delta)$. Besides, we assume that*
(a) $1 \le p_1 \le p_2 \le \infty$ *or* $\tilde{p} < p_2 < p_1 \le \infty$,
(b) $\delta \ne \alpha$,
(c) $p_2 < \infty$ *when* $p_1 < p_2$.

Denote by d_n the nth Kolmogorov number of the Sobolev embedding

$$A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d). \tag{2.1}$$

Then

$$d_n \sim n^{-\varkappa},$$

where

(i) $\varkappa = \frac{\mu}{d}$ *if* $1 \le p_1 \le p_2 \le 2$ *or* $2 < p_1 = p_2 \le \infty$,
(ii) $\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2}$ *if* $\tilde{p} < p_2 < p_1 \le \infty$,
(iii) $\varkappa = \frac{\mu}{d} + \frac{1}{2} - \frac{1}{p_2}$ *if* $1 \le p_1 < 2 < p_2 < \infty$ *and* $\mu > \frac{d}{p_2}$,
(iv) $\varkappa = \frac{\mu}{d} \cdot \frac{p_2}{2}$ *if* $1 \le p_1 < 2 < p_2 < \infty$ *and* $\mu < \frac{d}{p_2}$,
(v) $\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2}$ *if* $2 \le p_1 < p_2 < \infty$ *and* $\mu > \frac{d}{p_2}\theta$,
(vi) $\varkappa = \frac{\mu}{d} \cdot \frac{p_2}{2}$ *if* $2 \le p_1 < p_2 < \infty$ *and* $\mu < \frac{d}{p_2}\theta$.

Remark 2.6. *Similar conclusions on the nth Kolmogorov number could be made for Corollary 19 in Skrzypczak [24] with its corrigendum to part (iv) given in [26]. Of course, the counterexample to our new part (iv) could be also made for the limiting case $\delta = \frac{d}{p_2}$ by virtue of the special example appeared at the end of [26].*

For $1 \le p \le \infty$, we set

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty, \\ \infty & \text{if } p = 1. \end{cases}$$

Theorem 2.7. *Suppose* $1 \le p_1, p_2, q_1, q_2 \le \infty$ *and* $-\infty < s_2 < s_1 < \infty$. *Let* $\alpha > 0$, $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$, $\theta_1 = \frac{1/p_1 - 1/p_2}{1/p_1 - 1/2}$, *and* $\frac{1}{\tilde{p}} = \frac{\mu}{d} + \frac{1}{p_1}$, *where* $\mu = \min(\alpha, \delta)$. *Besides, we assume that*
(a) $1 \le p_1 \le p_2 \le \infty$ *or* $\tilde{p} < p_2 < p_1 \le \infty$,
(b) $\delta \ne \alpha$,
(c) $p_1 > 1$ *when* $p_1 < p_2$.

Denote by c_n *the nth Gelfand number of the Sobolev embedding (2.1). Then*
$$c_n \sim n^{-\varkappa},$$
where

(i) $\varkappa = \frac{\mu}{d}$ *if* $2 \le p_1 \le p_2 \le \infty$ *or* $1 \le p_1 = p_2 < 2$,

(ii) $\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2}$ *if* $\tilde{p} < p_2 < p_1 \le \infty$,

(iii) $\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{2}$ *if* $1 < p_1 < 2 < p_2 \le \infty$ *and* $\mu > \frac{d}{p_1'}$,

(iv) $\varkappa = \frac{\mu}{d} \cdot \frac{p_1'}{2}$ *if* $1 < p_1 < 2 < p_2 \le \infty$ *and* $\mu < \frac{d}{p_1'}$,

(v) $\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2}$ *if* $1 < p_1 < p_2 \le 2$ *and* $\mu > \frac{d}{p_1'}\theta_1$,

(vi) $\varkappa = \frac{\mu}{d} \cdot \frac{p_1'}{2}$ *if* $1 < p_1 < p_2 \le 2$ *and* $\mu < \frac{d}{p_1'}\theta_1$.

Remark 2.8. *In the above two theorems, the two function spaces in the embedding (2.1) may be of different types, i.e., one is the Besov space, and the other is the* $F^s_{p,q}$*-space.*

Remark 2.9. *From Definition 2.1, we know that an operator* $f \mapsto wf$ *is an isomorphic mapping from* $B^s_{p,q}(\mathbb{R}^d, w)$ *onto* $B^s_{p,q}(\mathbb{R}^d)$ *(similarly in the F-case). So by Remark 1.1, we get*

$$s_n(\mathrm{id},\ B^{s_1}_{p_1,q_1}(\mathbb{R}^d, v_1),\ B^{s_2}_{p_2,q_2}(\mathbb{R}^d, v_2)) \ \sim\ s_n(\mathrm{id},\ B^{s_1}_{p_1,q_1}(\mathbb{R}^d, v_1/v_2),\ B^{s_2}_{p_2,q_2}(\mathbb{R}^d)),$$

where s_n *denotes either of the two quantities* c_n *or* d_n*, (similarly in the F-case and even in the general case emphasized in Remark 2.8). Therefore, without loss of generality we may assume that the target space is an unweighted space.*

Remark 2.10. *For the limiting case* $\delta = \alpha$*, the exact order of related n-widths may possibly depend on* q_1 *and* q_2*. In general, the discretization method is not perfect for this case. There are partial results on approximation numbers in [18, 25] and two-sided estimates with minor gaps in [10]. Some ideas from [14, 15] may also be helpful to further research in this situation.*

Now, we wish to compare the approximation, Gelfand and Kolmogorov numbers of the Sobolev embedding (2.1). First, recall some basic facts about approximation numbers. We define the nth approximation number of T by

$$a_n(T) = \inf\{\|T - L\| :\ L \in \mathcal{L}(X, Y),\ \mathrm{rank}(L) < n\}, \quad n \in \mathbb{N}, \tag{2.2}$$

where $\mathrm{rank}(L)$ denotes the dimension of $L(X)$. We refer to [5, 19, 21] for detailed discussions of this concept and further references. Let us mention that the approximation numbers are the largest among all s-numbers. There exist the following relationships:

$$c_n(T) \le a_n(T), \quad d_n(T) \le a_n(T), \quad n \in \mathbb{N}. \tag{2.3}$$

We again assume that $1 \le p_1, p_2, q_1, q_2 \le \infty$, $-\infty < s_2 < s_1 < \infty$, $\alpha > 0$, $\delta = s_1 - s_2 - d(\frac{1}{p_1} - \frac{1}{p_2}) > 0$, $\delta \neq \alpha$, $\mu = \min(\alpha, \delta)$, and $\frac{1}{\tilde{p}} = \frac{\mu}{d} + \frac{1}{p_1}$. We would like to discuss, when $a_n \sim c_n$, $a_n \sim d_n$, or $c_n \sim d_n$ holds true for the Sobolev embedding (2.1). The comparison of our results with the known results for the approximation numbers from [24] shows that

(i) $a_n \sim c_n$ if either

- (a) $2 \le p_1 < p_2 \le \infty$ or,
- (b) $\tilde{p} < p_2 \le p_1 \le \infty$ or,
- (c) $1 < p_1 < p_1' \le p_2 \le \infty$ and $\mu \neq \frac{d}{p_1'}$;

(ii) $a_n \sim d_n$ if either

- (a) $1 \le p_1 < p_2 \le 2$ or,
- (b) $\tilde{p} < p_2 \le p_1 \le \infty$ or,
- (c) $1 \le p_1 < 2 < p_2 \le p_1' \le \infty$, $p_2 < \infty$ and $\mu \neq \frac{d}{p_2}$;

(iii) $c_n \sim d_n$ if either

- (a) $\tilde{p} < p_2 \le p_1 \le \infty$ or,
- (b) $1 < p_1 < p_1' = p_2 < \infty$ and $\mu \neq \frac{d}{p_2}$.

Remark 2.11. *In the case of more general weight classes, we will show some asymptotic estimates of the approximation, Gelfand and Kolmogorov numbers of the corresponding embeddings in a forthcoming paper based on ideas from T. Kühn et al. [16].*

3 Sequence spaces and Kolmogorov numbers

3.1 Discretization of function spaces

There are various ways to associate to a Besov space a certain sequence space. Here we are going to use the discrete wavelet transform, a well-developed method of discretization, see [14] where quasi-Banach case is included, and [29] for a survey.

Let $\widetilde{\varphi}$ be an orthogonal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity and let $\widetilde{\psi}$ be a corresponding wavelet. Then the tensor product gives a scaling function φ and associated wavelets $\psi_1, \ldots, \psi_{2^d-1}$, all defined on $\mathbb{R}^d$. More exactly, we suppose

$$\widetilde{\varphi} \in C^r(\mathbb{R}) \quad \text{and} \quad \operatorname{supp} \widetilde{\varphi} \subset [-N_1, N_1]$$

for some $r \in \mathbb{N}$ and $N_1 > 0$. Then we have

$$\varphi, \psi_i \in C^r(\mathbb{R}^d) \quad \text{and} \quad \operatorname{supp} \varphi, \operatorname{supp} \psi_i \subset [-N_2, N_2]^d, \quad i = 1, \ldots, 2^d - 1. \tag{3.1}$$

We shall use the standard abbreviations

$$\varphi_{j,k}(x) = 2^{jd/2} \varphi(2^j x - k) \quad \text{and} \quad \psi_{i,j,k} = 2^{jd/2} \psi_i(2^j x - k), \tag{3.2}$$

where $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}^d$.

Proposition 3.1. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Assume*

$$r > \max(s, \frac{2d}{p} + \frac{d}{2} - s).$$

Then a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^s_{p,q}(w_\alpha)$ if and only if

$$\begin{aligned} \|f|B^s_{p,q}(w_\alpha)\|^{\clubsuit} = &\Big(\sum_{k\in\mathbb{Z}^d} |\langle f, \varphi_{0,k}\rangle w_\alpha(k)|^p\Big)^{1/p} \\ &+ \sum_{i=1}^{2^d-1} \Big\{\sum_{j=0}^{\infty} 2^{j\left(s+d(\frac{1}{2}-\frac{1}{p})\right)q} \Big(\sum_{k\in\mathbb{Z}^d} |\langle f, \psi_{i,j,k}\rangle w_\alpha(2^{-j}k)|^p\Big)^{q/p}\Big\}^{1/q} < \infty. \end{aligned} \tag{3.3}$$

Moreover, $\|f|B^s_{p,q}(w_\alpha)\|^{\clubsuit}$ may be used as an equivalent norm in $B^s_{p,q}(w_\alpha)$.

Remark 3.2. *The proof of this proposition may be found in Haroske and Triebel [14]. One can also consult [15] for historical remarks.*

Let $1 \le p, q \le \infty$. Inspired by Proposition 3.1 we will work with the following weighted sequence spaces

$$\begin{aligned} \ell_q(2^{js}\ell_p(\alpha)) := \Big\{ &\lambda = (\lambda_{j,k})_{j,k} : \quad \lambda_{j,k} \in \mathbb{C}, \\ &\|\lambda|\ell_q(2^{js}\ell_p(\alpha))\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\Big(\sum_{k\in\mathbb{Z}^d} |\lambda_{j,k}\, w_{j,k}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}, \end{aligned} \tag{3.4}$$

(usual modification if $p = \infty$ and/or $q = \infty$), where $w_{j,k} = w_\alpha(2^{-j}k)$. If $s = 0$ we will write $\ell_q(\ell_p(\alpha))$. In contrast to the norm defined in (3.3), the finite summation on $i = 1, 2, \ldots, 2^d - 1$ is irrelevant and can be omitted.

3.2 Kolmogorov numbers of embeddings of some sequence spaces

To begin with, we shall recall some lemmata. Lemma 3.3 follows trivially from results of Gluskin [8] and Edmunds and Triebel [5].

Lemma 3.3. *Let $N \in \mathbb{N}$.*

(i) *If $1 \le p_1 \le p_2 \le 2$ and $n \le \frac{N}{4}$, then*

$$d_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim 1.$$

(ii) *If $1 \le p_1 < 2 < p_2 < \infty$ and $n \le \frac{N}{4}$, then*

$$d_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim \min\{1, N^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}.$$

(iii) *If $2 < p_1 = p_2 \le \infty$ and $n \le N$, then*

$$d_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim 1.$$

(iv) *If $2 \le p_1 < p_2 < \infty$ and $n \le N$, then*

$$d_n\left(\mathrm{id}, \ell_{p_1}^N, \ell_{p_2}^N\right) \sim \xi^\theta,$$

where $\xi = \min\{1, N^{\frac{1}{p_2}} n^{-\frac{1}{2}}\}$, $\theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2}$.

For $p_2 < p_1$ the corresponding Kolmogorov numbers can be calculated trivially by virtue of Pietsch [19, 20], or Pinkus [21, p. 203].

Lemma 3.4. *Let $1 \le p_2 < p_1 \le \infty$ and $n \le N$. Then*

$$d_n\left(\mathrm{id}, \ell_{p_1}^N, \ell_{p_2}^N\right) = (N - n + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

The following assertion is a simple corollary of Lemma 3.3. And the proof is the same as that of Lemma 10 in Skrzypczak [24].

Lemma 3.5. *Suppose $1 \le p_1 < 2 < p_2 < \infty$ and $N \in \mathbb{N}$. Then there is a positive constant C independent of N and n such that*

$$d_n\left(\mathrm{id}, \ell_{p_1}^N, \ell_{p_2}^N\right) \le C \begin{cases} 1 & \text{if } n \le N^{\frac{2}{p_2}}, \\ N^{\frac{1}{p_2}} n^{-\frac{1}{2}} & \text{if } N^{\frac{2}{p_2}} < n \le N, \\ 0 & \text{if } n > N. \end{cases} \tag{3.5}$$

Proposition 3.6. *Suppose $1 \le p_1 < 2 < p_2 < \infty$ and $\delta \ne \alpha$. We set*

$$\varkappa = \begin{cases} \frac{\mu}{d} + \frac{1}{2} - \frac{1}{p_2} & \text{if } \mu > \frac{d}{p_2}, \\ \frac{\mu}{d} \cdot \frac{p_2}{2} & \text{if } \mu < \frac{d}{p_2}, \end{cases} \tag{3.6}$$

where $\mu = \min(\alpha, \delta)$. Then

$$d_n\Big(\mathrm{id}, \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)), \ell_{q_2}(\ell_{p_2})\Big) \sim n^{-\varkappa}. \tag{3.7}$$

Proof. By Lemma 3.3 and Lemma 3.5, the proof of the proposition can be finished in the same manner as in the proof of Prop. 11 in [24] with the important complement given in [26]. The only change is that $t = \min(p_1', p_2)$ is replaced by p_2 in our proof. $\square$

Proposition 3.7. *Suppose $2 \le p_1 < p_2 < \infty$ and $\delta \ne \alpha$. We set*

$$\varkappa = \begin{cases} \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2} & \text{if } \mu > \frac{d}{p_2}\theta, \\ \frac{\mu}{d} \cdot \frac{p_2}{2} & \text{if } \mu < \frac{d}{p_2}\theta, \end{cases} \tag{3.8}$$

where $\mu = \min(\alpha, \delta)$, $\theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2}$. Then

$$d_n\Big(\mathrm{id}, \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)), \ell_{q_2}(\ell_{p_2})\Big) \sim n^{-\varkappa}. \tag{3.9}$$

Proof. `Step 1`. Preparations. We denote

$$\Lambda := \{\lambda = (\lambda_{j,k}) : \quad \lambda_{j,k} \in \mathbb{C}, \quad j \in \mathbb{N}_0,\, k \in \mathbb{Z}^d\},$$

and set

$$B_1 = \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)) \quad \text{and} \quad B_2 = \ell_{q_2}(\ell_{p_2}).$$

Let $I_{j,i} \subset \mathbb{N}_0 \times \mathbb{Z}^d$ be such that

$$I_{j,0} := \{(j,k) :\, |k| \le 2^j\}, \quad j \in \mathbb{N}_0, \tag{3.10}$$

$$I_{j,i} := \{(j,k) :\, 2^{j+i-1} < |k| \le 2^{j+i}\}, \quad i \in \mathbb{N}, \quad j \in \mathbb{N}_0. \tag{3.11}$$

Besides, let $P_{j,i} : \Lambda \mapsto \Lambda$ be the canonical projection onto the coordinates in $I_{j,i}$; i.e., for $\lambda \in \Lambda$, we set

$$(P_{j,i}\lambda)_{u,v} := \begin{cases} \lambda_{u,v} & (u,v) \in I_{j,i}, \\ 0 & \text{otherwise}, \end{cases} \quad u \in \mathbb{N}_0, \quad v \in \mathbb{Z}^d, \quad i \ge 0.$$

Then

$$M_{j,i} := |I_{j,i}| \sim 2^{d(j+i)}, \tag{3.12}$$

$$\mathrm{id}_\Lambda = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P_{j,i}, \tag{3.13}$$

$$w_{j,k} = w_\alpha(2^{-j}k) \sim 2^{\alpha i} \quad \text{if} \quad (j,k) \in I_{j,i}, \quad i \ge 0. \tag{3.14}$$

Due to simple monotonicity arguments and explicit properties of the Kolmogorov numbers we have

$$\begin{aligned} d_n(P_{j,i}, B_1, B_2) &\le \frac{1}{\inf_{k \in I_{j,i}} w_{j,k}} 2^{-j\delta} d_n(\mathrm{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \\ &\le c 2^{-j\delta - i\alpha} d_n(\mathrm{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}). \end{aligned} \tag{3.15}$$

`Step 2`. The operator ideal comes into play. Under the assumption $2 \le p_1 < p_2 < \infty$, it is easy to prove that $0 < \theta \le 1$. To shorten notations we shall put $\tau = \frac{p_2}{\theta}$, $h = \frac{2}{\theta}$, and $\frac{1}{s} = \frac{1}{\gamma} + \frac{1}{h}$ for any $s > 0$. In terms of (1.9) and (3.15), we have

$$L_{s,\infty}^{(d)}(P_{j,i}) \le c 2^{-j\delta - i\alpha} L_{s,\infty}^{(d)}(\mathrm{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}). \tag{3.16}$$

The known asymptotic behavior of the Kolmogorov numbers $d_n(\mathrm{id}, \ell_{p_1}^N, \ell_{p_2}^N)$, cf. Lemma 3.3 (iv), and (3.12) yield that

$$L_{h,\infty}^{(d)}(\mathrm{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \le C 2^{d(j+i)/\tau}, \tag{3.17}$$

$$L_{s,\infty}^{(d)}(\mathrm{id}, \ell_{p_1}^{M_{j,i}}, \ell_{p_2}^{M_{j,i}}) \le C 2^{d(j+i)(\frac{1}{\tau}+\frac{1}{\gamma})} \qquad \text{if} \quad \frac{1}{s} > \frac{1}{h}. \tag{3.18}$$

`Step 3`. The estimate of $d_n(\mathrm{id}, B_1, B_2)$ from above in the first case $\mu > \frac{d}{\tau}$. For any given $M \in \mathbb{N}_0$, we put

$$P := \sum_{m=0}^{M} \sum_{j+i=m} P_{j,i} \qquad \text{and} \qquad Q := \sum_{m=M+1}^{\infty} \sum_{j+i=m} P_{j,i}. \tag{3.19}$$

`Substep 3.1.` Estimate of $d_n(P, B_1, B_2)$. Let $\frac{1}{s} > \frac{1}{h}$. Then in view of (1.10), (3.16) and (3.18), we have

$$\begin{aligned} L^{(d)}_{s,\infty}(P)^\rho &\le \sum_{m=0}^{M} \sum_{j+i=m} L^{(d)}_{s,\infty}(P_{j,i})^\rho \\ &\le c_1 \sum_{m=0}^{M} \sum_{j+i=m} 2^{-\rho(j\delta+i\alpha)} 2^{\rho md(\frac{1}{\tau}+\frac{1}{\gamma})} \\ &\le c_2 \sum_{m=0}^{M} 2^{\rho md(\frac{1}{\tau}+\frac{1}{\gamma}-\frac{\mu}{d})}. \end{aligned} \tag{3.20}$$

In the last inequality we used our assumption $\delta \neq \alpha$. We choose γ such that $d(\frac{1}{\tau} + \frac{1}{\gamma}) - \mu > 0$. Then (3.20) yields

$$L^{(d)}_{s,\infty}(P) \le c\ 2^{dM(\frac{1}{\tau}+\frac{1}{\gamma}-\frac{\mu}{d})}. \tag{3.21}$$

Using (1.9) and (3.21), we get

$$d_{2^{dM}}(P, B_1, B_2) \le c_3 2^{dM(\frac{1}{\tau}-\frac{1}{h}-\frac{\mu}{d})}. \tag{3.22}$$

We put $n = 2^{Md}$. Then

$$d_n(P, B_1, B_2) \le c_3 n^{\frac{1}{\tau}-\frac{1}{h}-\frac{\mu}{d}} = c_3 n^{-(\frac{\mu}{d}+\frac{1}{p_1}-\frac{1}{p_2})}. \tag{3.23}$$

`Substep 3.2.` Estimate of $d_n(Q, B_1, B_2)$. In a similar way to (3.20), we obtain by $\delta \neq \alpha$ and (3.17) that

$$L^{(d)}_{h,\infty}(Q)^\rho \le c_1 \sum_{m=M+1}^{\infty} 2^{\rho md(\frac{1}{\tau}-\frac{\mu}{d})}. \tag{3.24}$$

Since $\mu > \frac{d}{\tau}$, we have

$$L^{(d)}_{h,\infty}(Q) \le c 2^{dM(\frac{1}{\tau}-\frac{\mu}{d})}. \tag{3.25}$$

By virtue of (1.9), we have

$$d_{2^{dM}}(Q, B_1, B_2) \le c_2 2^{dM(\frac{1}{\tau}-\frac{1}{h}-\frac{\mu}{d})}. \tag{3.26}$$

Take $n = 2^{Md}$. Then

$$d_n(Q, B_1, B_2) \le c_2 n^{\frac{1}{\tau}-\frac{1}{h}-\frac{\mu}{d}} = c_2 n^{-(\frac{\mu}{d}+\frac{1}{p_1}-\frac{1}{p_2})}. \tag{3.27}$$

`Substep 3.3.` Under the assumption $\mu > \frac{d}{\tau}$, the estimate from above follows from (3.23) and (3.27), by means of the inequality below,

$$d_{2n}(\mathrm{id}, B_1, B_2) \le d_n(P, B_1, B_2) + d_n(Q, B_1, B_2). \tag{3.28}$$

`Step 4.` The estimate of $d_n(\mathrm{id}, B_1, B_2)$ from above in the second case $\mu < \frac{d}{\tau}$. Inspired by [26], we use the following division

$$\mathrm{id} = \sum_{m=0}^{M_1} \sum_{j+i=m} P_{j,i} + \sum_{m=M_1+1}^{M_2} \sum_{j+i=m} P_{j,i} + \sum_{m=M_2+1}^{\infty} \sum_{j+i=m} P_{j,i}, \tag{3.29}$$

where $M_1, M_2 \in \mathbb{N}$ and $M_1 < M_2$, which will be determined later on. Using the subadditivity of s-numbers, we have

$$d_{n'}(\mathrm{id}, B_1, B_2) \le \triangle_1 + \triangle_2 + \triangle_3, \tag{3.30}$$

where

$$\triangle_1 = \sum_{m=0}^{M_1} \sum_{j+i=m} d_{n_{j,i}}(P_{j,i}), \qquad \triangle_2 = \sum_{m=M_1+1}^{M_2} \sum_{j+i=m} d_{n_{j,i}}(P_{j,i}),$$
$$\triangle_3 = \sum_{m=M_2+1}^{\infty} \sum_{j+i=m} \|P_{j,i}\|, \qquad n' - 1 = \sum_{m=0}^{M_2} \sum_{j+i=m} (n_{j,i} - 1).$$

Note that for $\triangle_3$, we have $j + i > M_2$ in the sum, and we take $n_{j,i} = 1$. Now let $n \in \mathbb{N}$ be given. We take

$$M_1 = \left[\frac{\log_2 n}{d} - \frac{\log_2 \log_2 n}{d}\right] \qquad \text{and} \qquad M_2 = \left[\frac{\tau}{h} \cdot \frac{\log_2 n}{d}\right],$$

where $[a]$ denotes the largest integer smaller than $a \in \mathbb{R}$ and $\log_2 n$ is a dyadic logarithm of n. Then

$$\begin{aligned} \triangle_3 &= \sum_{m=M_2+1}^{\infty} \sum_{j+i=m} \|P_{j,i}\| \le c_1 \sum_{m=M_2+1}^{\infty} \sum_{j+i=m} 2^{-j\delta} 2^{-i\alpha} \\ &\le c_2 \sum_{m=M_2+1}^{\infty} 2^{-m\mu} \le c_3 2^{-M_2\mu} \le c_3 n^{-\varkappa}. \end{aligned}$$

Next, we choose proper $n_{j,i}$ for estimating $\triangle_1$ and $\triangle_2$. If $i + j \le M_1$, we take $n_{j,i} = M_{j,i} + 1$ such that $d_{n_{j,i}}(P_{j,i}) = 0$ and $\triangle_1 = 0$. And we obtain

$$\sum_{m=0}^{M_1} \sum_{j+i=m} n_{j,i} \le c_1 \sum_{m=0}^{M_1} (m+1) 2^{md} \le c_2 M_1 \cdot 2^{M_1 d} \le c_3 n.$$

Now we give the crucial choice of $n_{j,i}$ for the second sum $\triangle_2$. We take

$$n_{j,i} = [n^{1-\varepsilon} \cdot 2^{iz_1} \cdot 2^{jz_2}],$$

where ε, z_1, z_2 are positive real numbers such that

$$\alpha + \frac{z_1}{h} < \frac{d}{\tau}, \quad 0 < \frac{z_1 - z_2}{h} < \delta - \alpha \quad \text{and} \quad \frac{z_1 \tau}{hd} = \varepsilon \quad \text{if } \delta > \alpha,$$

or

$$\delta + \frac{z_2}{h} < \frac{d}{\tau}, \quad 0 < \frac{z_2 - z_1}{h} < \alpha - \delta \quad \text{and} \quad \frac{z_2 \tau}{hd} = \varepsilon \quad \text{if } \delta < \alpha.$$

Note that the relation, $0 < \varepsilon < 1$, holds obviously. Then

$$\sum_{m=M_1+1}^{M_2} \sum_{j+i=m} n_{j,i} \le c_1 n^{1-\varepsilon} \sum_{m=M_1+1}^{M_2} 2^{m \cdot \max(z_1, z_2)} \le c_2 n^{1-\varepsilon} \cdot n^{\frac{\tau}{hd} \max(z_1, z_2)} = c_2 n,$$

and, in terms of (3.17),

$$
\begin{aligned}
\sum_{m=M_1+1}^{M_2} \sum_{j+i=m} d_{n_{j,i}}(P_{j,i}) &\le c_1 \sum_{m=M_1+1}^{M_2} \sum_{j+i=m} 2^{-j\delta - i\alpha} 2^{(i+j)d/\tau} [n^{1-\varepsilon} \cdot 2^{iz_1} \cdot 2^{jz_2}]^{-\frac{1}{h}} \\
&\le c_2 n^{-\frac{1}{h}(1-\varepsilon)} \sum_{m=M_1+1}^{M_2} 2^{md/\tau} 2^{-m\cdot \min(\alpha+\frac{z_1}{h}, \delta+\frac{z_2}{h})} \\
&\le c_3 n^{-\frac{1}{h}(1-\varepsilon)} n^{\frac{1}{h}} n^{-\frac{\tau}{hd}\min(\alpha+\frac{z_1}{h}, \delta+\frac{z_2}{h})} \\
&= c_3 n^{\frac{\varepsilon}{h} - \frac{\tau}{hd}\min(\alpha+\frac{z_1}{h}, \delta+\frac{z_2}{h})} \\
&= c_3 n^{-\frac{\tau\mu}{hd}} = c_3 n^{-\varkappa}.
\end{aligned}
$$

Hence the estimate from above in the second case is finished.
`Step 5.` The estimate of $d_n(\mathrm{id}, B_1, B_2)$ from below. Consider the following diagram

$$
\begin{tikzcd}
\ell_{p_1}^{M_{j,i}} \arrow[r, "S_{j,i}"] \arrow[d, "\mathrm{id}_1"] & \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)) \arrow[d, "\mathrm{id}"] \\
\ell_{p_2}^{M_{j,i}} & \ell_{q_2}(\ell_{p_2}) \arrow[l, "T_{j,i}"']
\end{tikzcd}
\tag{3.31}
$$

Here,

$$
\begin{aligned}
(S_{j,i}\eta)_{u,v} &:= \begin{cases} \eta_{\varphi(u,v)} & \text{if} \quad (u,v) \in I_{j,i}, \\ 0 & \text{otherwise}, \end{cases} \\
(T_{j,i}\lambda)_{\varphi(u,v)} &:= \lambda_{u,v}, \qquad\qquad (u,v) \in I_{j,i},
\end{aligned}
$$

and φ denotes a bijection of $I_{j,i}$ onto $\{1, \dots, M_{j,i}\}$, $j \in \mathbb{N}_0$, $i \in \mathbb{N}_0$; cf. (3.10) and (3.11). Observe that

$$
\begin{aligned}
S_{j,i} &\in \mathcal{L}\left(\ell_{p_1}^{M_{j,i}},\, \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha))\right) \qquad \text{and} \quad \|S_{j,i}\| = 2^{j\delta+i\alpha}, \\
T_{j,i} &\in \mathcal{L}\left(\ell_{q_2}(\ell_{p_2}),\, \ell_{p_2}^{M_{j,i}}\right) \qquad \text{and} \quad \|T_{j,i}\| = 1.
\end{aligned}
$$

Hence we obtain

$$
d_n(\,\mathrm{id}_1) \le \|S_{j,i}\| \, \|T_{j,i}\| \, d_n(\,\mathrm{id}). \tag{3.32}
$$

(i) Let $\frac{d}{\tau} < \delta \le \alpha$. We consider $N := M_{j,0} = |I_{j,0}| \sim 2^{dj}$, $j \ge \frac{2}{d}$. Then

$$
\|S_{j,0}\| \le C 2^{j\delta} \quad \text{and} \quad \|T_{j,0}\| = 1.
$$

Put $m = \frac{N}{4} \sim 2^{jd-2}$. And for sufficiently large N we have $m \ge N^{\frac{2}{p_2}}$ since $p_2 > 2$. Consequently,

$$
d_m(\mathrm{id}_1,\, \ell_{p_1}^N,\, \ell_{p_2}^N) \sim \left(N^{\frac{1}{p_2}} m^{-\frac{1}{2}}\right)^{\theta} \sim 2^{\theta(jd-2)(\frac{1}{p_2}-\frac{1}{2})} \sim 2^{(jd-2)(\frac{1}{p_2}-\frac{1}{p_1})}.
$$

Using (3.32), we obtain

$$
d_{2^{jd-2}}(\mathrm{id}) \ge C_1 2^{-j\delta} 2^{(jd-2)(\frac{1}{p_2}-\frac{1}{p_1})} \ge C_2 2^{(jd-2)(\frac{1}{p_2}-\frac{1}{p_1}-\frac{\delta}{d})}.
$$

Then the monotonicity of the Kolmogorov numbers implies that for any $n \in \mathbb{N}$

$$d_n(\mathrm{id}) \geq C_3 n^{-(\frac{\delta}{d}+\frac{1}{p_1}-\frac{1}{p_2})}. \tag{3.33}$$

(ii) Let $\frac{d}{\tau} < \alpha < \delta$. We consider $N := M_{0,i} = |I_{0,i}| \sim 2^{di}$, $i \geq \frac{2}{d}$. Then

$$\|S_{0,i}\| \leq C2^{i\alpha} \quad \text{and} \quad \|T_{0,i}\| = 1.$$

Also put $m = \frac{N}{4} \sim 2^{di-2}$. Hence we have similarly for any $n \in \mathbb{N}$

$$d_n(\mathrm{id}) \geq C n^{-(\frac{\alpha}{d}+\frac{1}{p_1}-\frac{1}{p_2})}. \tag{3.34}$$

(iii) Let $\delta \leq \frac{d}{\tau}$ and $\delta < \alpha$. We select the same N, S, and T as in point (i) and take $m = \left[N^{\frac{2}{p_2}}\right] \leq \frac{N}{4}$ for sufficiently large N. Then $N^{\frac{1}{p_2}} m^{-\frac{1}{2}} \sim 1$. Hence by Lemma 3.3 and (3.32) we obtain

$$d_m(\mathrm{id}) \geq C2^{-j\delta} = C2^{-jd\frac{2}{p_2}\frac{p_2\delta}{2d}},$$

and then for any $n \in \mathbb{N}$

$$d_n(\mathrm{id}) \geq C n^{-\frac{p_2\delta}{2d}}. \tag{3.35}$$

(iv) Let $\alpha \leq \frac{d}{\tau}$ and $\alpha \leq \delta$. We select the same N, S, and T as in point (ii) and take $m = \left[N^{\frac{2}{p_2}}\right]$ in the same way as in point (iii). Then analogously

$$d_m(\mathrm{id}) \geq C2^{-i\alpha} = C2^{-di\frac{2}{p_2}\frac{p_2\alpha}{2d}},$$

and in consequence, for any $n \in \mathbb{N}$

$$d_n(\mathrm{id}) \geq C n^{-\frac{p_2\alpha}{2d}}. \tag{3.36}$$

The proof of the proposition is now complete. $\square$

Proposition 3.8. *Suppose* $1 \leq p_1 \leq p_2 \leq 2$ *or* $2 < p_1 = p_2 \leq \infty$ *and* $\delta \neq \alpha$. *We set*

$$\varkappa = \frac{\mu}{d}, \qquad \textit{where} \quad \mu = \min(\alpha, \delta). \tag{3.37}$$

Then

$$d_n\left(\mathrm{id}, \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)), \ell_{q_2}(\ell_{p_2})\right) \sim n^{-\varkappa}. \tag{3.38}$$

In view of Lemma 3.3, the proof of this proposition follows exactly as in the proof of Prop. 13 in [24].

Proposition 3.9. *Suppose* $1 \leq p_1, p_2 \leq \infty$, $\frac{1}{\tilde{p}} = \frac{\min(\alpha,\delta)}{d} + \frac{1}{p_1}$, *and* $\delta \neq \alpha$. *Assume* $\tilde{p} < p_2 < p_1 \leq \infty$, *and set*

$$\varkappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{p_2}, \qquad \textit{where} \quad \mu = \min(\alpha, \delta). \tag{3.39}$$

Then

$$d_n\left(\mathrm{id}, \ell_{q_1}(2^{j\delta}\ell_{p_1}(\alpha)), \ell_{q_2}(\ell_{p_2})\right) \sim n^{-\varkappa}. \tag{3.40}$$

By Lemma 3.4, the proof of this proposition can be finished in the same manner as in the proof of Prop. 15 in [24].

4 Proofs of the main results

4.1 Proof of Theorem 2.5

Based on Prop. 3.1, we now transfer the results of Subsection 3.2 for weighted sequence spaces to weighted function spaces.

First, for the embeddings given by (1.3), i.e., the Besov case, the assertions follow from Proposition 3.1 and 3.6-3.8, or Proposition 3.9, respectively.

For the general case, we estimate from above, by virtue of the multiplicativity property of Kolmogorov numbers, and the elementary embeddings below

$$A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_2}_{p_2,1}(\mathbb{R}^d) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d).$$

For the estimate from below we can consider the following embeddings

$$B^{s_1}_{p_1,1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,\infty}(\mathbb{R}^d).$$

$\square$

4.2 Proof of Theorem 2.7

We turn our attention to Gelfand numbers. First, we collect some necessary information on $c_n(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2})$, cf. [8, 19, 32], (1.6) and (1.7).

Lemma 4.1. *Let $N \in \mathbb{N}$.*

(i) *If $2 \le p_1 \le p_2 \le \infty$ and $n \le \frac{N}{4}$, then*

$$c_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim 1.$$

(ii) *If $1 < p_1 < 2 < p_2 \le \infty$ and $n \le \frac{N}{4}$, then*

$$c_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim \min\{1, N^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\}.$$

(iii) *If $1 \le p_1 = p_2 < 2$ and $n \le N$, then*

$$c_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim 1.$$

(iv) *If $1 < p_1 < p_2 \le 2$ and $n \le N$, then*

$$c_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) \sim \xi^{\theta_1},$$

where $\xi = \min\{1, N^{1-\frac{1}{p_1}} n^{-\frac{1}{2}}\}$, $\theta_1 = \frac{1/p_1 - 1/p_2}{1/p_1 - 1/2}$.

The proof of this lemma follows by (1.6), (1.7) and Lemma 3.3.

Lemma 4.2. *Let $1 \le p_2 < p_1 \le \infty$ and $n \le N$. Then*

$$c_n\left(\mathrm{id}, \ell^N_{p_1}, \ell^N_{p_2}\right) = (N - n + 1)^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

The proof of this lemma follows literally Pietsch [19, 20], and also Pinkus [21].

Now we are ready to prove Theorem 2.7. We can deal with the proof in a similar way to the one for Theorem 2.5, so we give the sketch here. For example, in the case (v), which is corresponding to the case (v) in Theorem 2.5, the proof can be based on Prop. 3.7. The changes begin with (3.15), where d_n is substituted by c_n. And the others go on trivially.

Remark 4.3. *For the quasi-Banach case,* $0 < p < 1$ or $0 < q < 1$, *the problem of these two quantities of the embeddings given by (2.1) becomes more complicated. Indeed, Lemma 3.3 and Lemma 3.4 can not be completely generalized to the quasi-Banach setting* $0 < p_1, p_2 \leq \infty$. *And the duality between these two quantities is not valid. Fortunately, some recent results provided by Foucart et al. [7] and Vybíral [32] are effective to a certain extent. This situation will be discussed in a forthcoming paper.*

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